On Taylor Model Methods for Solving Linear ODEs with Constant Coefficients

Markus Neher
Institut für Angewandte Mathematik
Universität Karlsruhe

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Introduction

IVP:

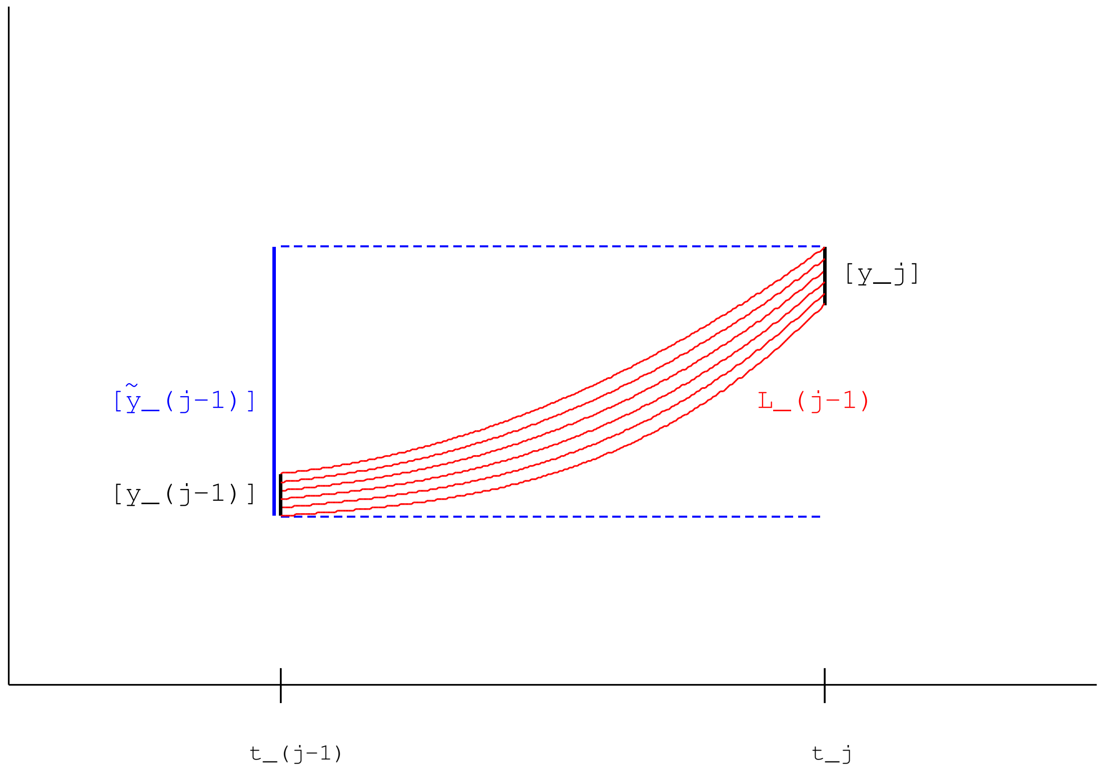
$$y' = B\,y$$

$$y(0) = y_0 = c_0 + C_0\,x \qquad (\ y(0) = y_0 \in [y_0]\ ),$$

where

- B: (n,n)-matrix,
- c_0: n-vector,
- C_0: (n,n)-diagonal matrix
- x: symbolic n-vector with components $x_i \in [-1,1]$.

Direct Interval Method



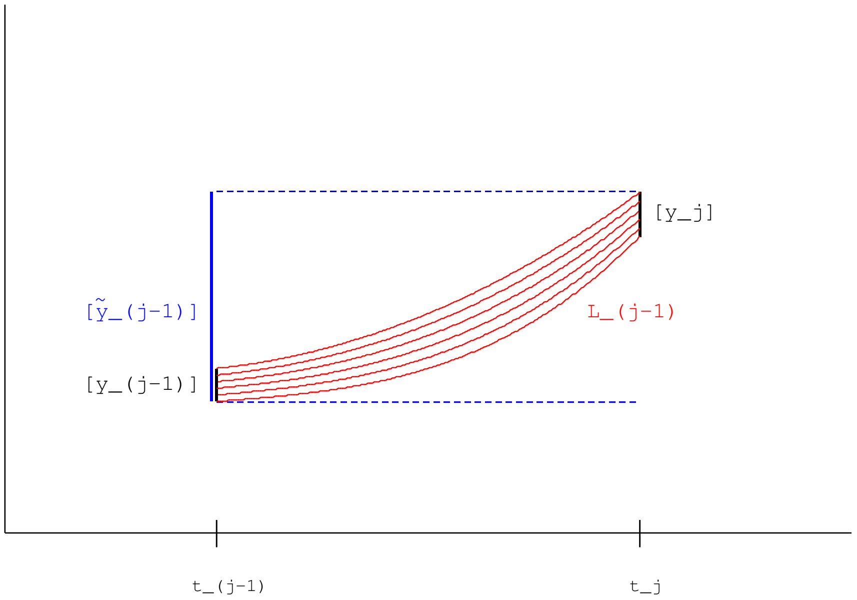
$$t_j := t_{j-1} + h, \quad j = 1, 2, \ldots.$$

$$L_{j-1} := \{u(t) \mid u' = Bu,\ u(t_{j-1}) = y_{j-1} \in [y_{j-1}],\ t \in [t_{j-1}, t_j]\}.$$

$[\tilde{y}_{j-1}] \in \mathbb{IR}^n$: a priori enclosure such that $\forall y \in L_{j-1}$:

$$y(t) \in [\tilde{y}_{j-1}] \text{ for all } t \in [t_{j-1}, t_j].$$

Direct Interval Method



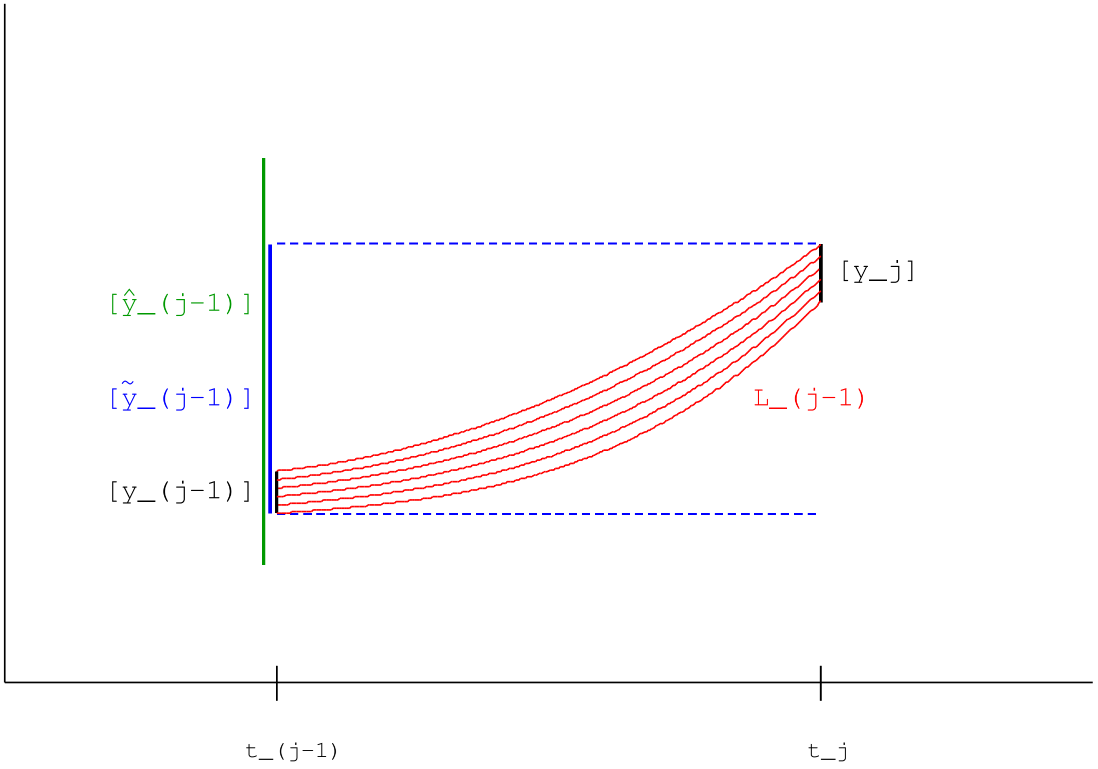
$$y(t_j) = Ty(t_{j-1}) + \tilde{z}_j,$$

where
$$T := \sum_{i=0}^{k} \frac{(hB)^i}{i!}, \qquad \tilde{z}_j \in [z_j] := \frac{(hB)^{k+1}}{(k+1)!}[\tilde{y}_{j-1}].$$

Direct interval method:

$$[y_j] := T[y_{j-1}] + \frac{(hB)^{k+1}}{(k+1)!}[\tilde{y}_{j-1}], \qquad j = 1, 2, \ldots .$$

Coarse Enclosure



Fixed point iteration: Determine $[\widehat{y}_{j-1}]$ such that

$$[y_{j-1}] + [0,h]\,B\,[\widehat{y}_{j-1}] \subseteq [\widehat{y}_{j-1}].$$

TM Method: Explicit Example

IVP:

$$y' = B\,y = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} y$$

$$y(0) = y_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \alpha, \beta \in [0,1]$$

Picard Iteration:

- $y_{1,0} := y_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
- $y_{1,1}(t) = y_0 + \int_0^t B\,y_{1,0}\,d\tau = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \int_0^t \begin{pmatrix} b_{11}\alpha + b_{12}\beta \\ b_{21}\alpha + b_{22}\beta \end{pmatrix} d\tau = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + tB \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
- $y_{1,2}(t) = y_0 + \int_0^t B\,y_{1,1}\,d\tau = (I + tB + \frac{(tB)^2}{2}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
- $y_{1,i}(t) = \sum_{\nu=0}^{i} \frac{(tB)^\nu}{\nu!} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad i = 0, 1, \ldots k$: $\qquad y_{1,k} = T\,y_0$

Self Inclusion

Let $I_0 := [0,0]$, $P_k(tB) := \sum_{\nu=0}^{k} \frac{(tB)^\nu}{\nu!}$.

Determine $\tilde{I}_1$ such that for some $t = h$:

$$y_0 + I_0 + \int_0^h B\left(P_k(\tau B)\, y_0 \; + I_0 + \tilde{I}_1\right) d\tau \;\subseteq\; P_k(hB)\, y_0 \; + I_0 + \tilde{I}_1 = T y_0 \; + I_0 + \tilde{I}_1$$

and let $I_1 := I_0 + \tilde{I}_1$.

Iteration:

$$y_j = T\, y_{j-1} + I_j, \quad j = 1, 2, \ldots,$$

that is

$$y_1 = T\, y_0 + I_1 = T\, c_0 + T C_0\, x + I_1$$

and

$$y_j = T^j\, c_0 + T^j C_0\, x + \sum_{i=1}^{j} T^{j-i} I_i.$$

Naive Taylor Model Method without Shrink Wrapping

- $C_0 = 0$: Point IVP $y' = By$, $y(t_0) = y_0 = c_0$:

  Naive TM method without SW coincides with the direct method.

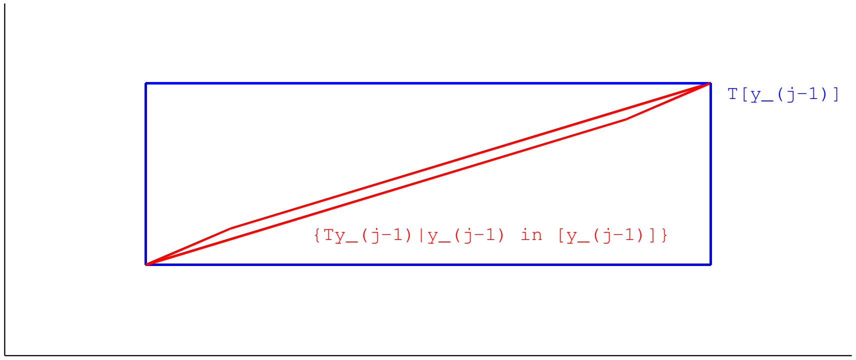
- $C_0 \neq 0$:

  Naive TM method without SW contains the same interval propagation of the local errors as the direct method.

  In particular, after j steps we have

$$w(T(T(\ldots T I_1)\ldots)) \;=\; |T|^{\,j-1} w(I_1).$$

Parallelepiped Method



$$\{Ty_{j-1} \mid y_{j-1} \in [y_{j-1}]\} \subseteq T[y_{j-1}]$$

Idea: Store $\{Ty_{j-1} \mid y_{j-1} \in [y_{j-1}]\}$ instead of $T[y_{j-1}]$.

Let $A_0 := I$, $\widehat{y}_0 := y_0$ and iterate:

$$\begin{aligned} s_j &:= m([z_j]), \quad \widehat{y}_j := T\widehat{y}_{j-1} + s_j \\ [y_j] &:= T\,\widehat{y}_{j-1} + (TA_{j-1})[r_{j-1}] + [z_j] \\ [r_j] &:= (A_j^{-1}TA_{j-1})[r_{j-1}] + A_j^{-1}([z_j] - s_j) \end{aligned}$$

Parallelepiped method:

$$A_j := TA_{j-1} \;\Rightarrow\; [r_j] := [r_{j-1}] + A_j^{-1}([z_j] - s_j).$$

Shrink Wrapping

Idea: Absorb the interval part of the TM into the polynomial part by increasing the polynomial coefficients.

Example: $$\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}+\begin{pmatrix}[-1,1]\\[1,5]\end{pmatrix}\mid\alpha,\beta\in[-1,1]\}$$

$$=\{\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}3&0\\0&3\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}\mid\alpha,\beta\in[-1,1]\}=\begin{pmatrix}[-2,4]\\[0,6]\end{pmatrix}$$

Shrink wrapping for ODE $y'=By$:

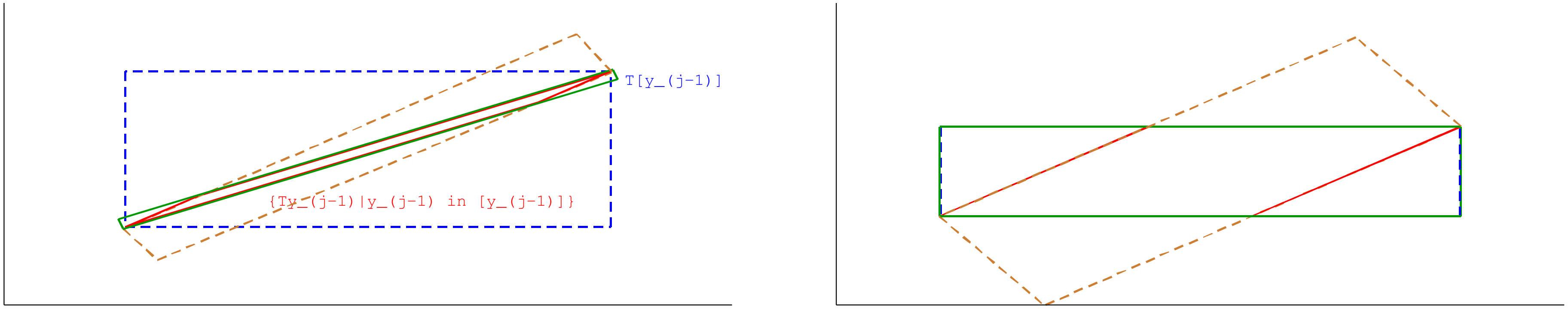
$$d_j := w\left((T^jC_{j-1})^{-1}I_j\right)/2,$$

$$C_j := (1+d_j)C_{j-1}, \quad I_j \;:=\; 0.$$

Thus, we obtain

$$y_j \;=\; T^j\,c_0+\prod_{i=1}^{j}(1+d_i)C_0\,x.$$

QR Method



Idea: Stabilize the iteration by orthogonalization

QR method: $TA_{j-1}P_j = Q_jR_j, \quad A_j := Q_j$

Preconditioning

Unconditioned TM: Flow of the ODE at $t = t_0$ is represented by a single TM:

$$y(t_0) = c_0 + C_0\, x = P_0 + I_0,$$

where

- c_0: n-vector,
- C_0: (n, n)-diagonal matrix,
- x: Symbolic n-vector with components $x_i \in [-1, 1]$,
- $P_0 = c_0 + C_0\, x$: Multivariate polynomial (of order 1),
- I_0: Zero interval.

Similar at $t = t_j$:

$$y(t_j) = c_j + C_j\, x + I_j = P_j + I_j$$

(linear ODE, hence linear set in x for all t).

Preconditioning

Preconditioned TM: Flow of the ODE at $t = t_j$ is represented by a concatenation of two TMs (and a scaling matrix S_j):

$$\begin{aligned} y(t_j) &= (P_{j,l} + I_{j,l}) \circ S_j \circ (P_{j,r} + I_{j,r}) \\ &= (c_{j,l} + C_{j,l}\,x + I_{j,l}) \circ S_j \circ (c_{j,r} + C_{j,r}\,x + I_{j,r}) \end{aligned}$$

Theorem: Integrating the flow of the ODE only acts on the left TM.

“Proof”: If

$$\int f(x,t)\,dt = F(x,t) \quad \text{and} \quad x = g(u),$$

then

$$\int f(g(u),t)\,dt = F(g(u),t).$$

Preconditioning

Preconditioning for linear ODEs: Flow at $t = t_j$:

$$y(t_j) = (c_{j,l} + C_{j,l}\, x + I_{j,l}) \circ S_j \circ (P_{j,r} + I_{j,r})$$

Flow after integration:

$$\begin{aligned} y(t_{j+1}) &= (Tc_{j,l} + TC_{j,l}\, x + I_{j+1,l}) \circ S_j \circ (P_{j,r} + I_{j,r}) \\ &= (Tc_{j,l} + C_{j+1,l}\, x + [0,0]) \\ &\quad \circ \Big\{ \Big[C_{j+1,l}^{-1} T C_{j,l}\, x + C_{j+1,l}^{-1} I_{j+1,l} \Big] \circ S_j \circ (P_{j,r} + I_{j,r}) \Big\} \\ &= (c_{j+1,l} + C_{j+1,l}\, x + [0,0]) \circ S_{j+1} \circ (P_{j+1,r} + I_{j+1,r}) \end{aligned}$$

The expression in brackets is the same as in the PE and QR interval iteration.

Preconditioning

Expression in brackets:

$$\left[C_{j+1,l}^{-1}TC_{j,l}\,x+C_{j+1,l}^{-1}I_{j+1,l}\right]\circ S_j\circ(c_{j,r}+C_{j,r}\,x+I_{j,r})$$

$$=C_{j+1,l}^{-1}TC_{j,l}S_jc_{j,r}+C_{j+1,l}^{-1}TC_{j,l}S_jC_{j,r}\,x+C_{j+1,l}^{-1}TC_{j,l}S_jI_{j,r}+C_{j+1,l}^{-1}I_{j+1,l}$$

$$=S_{j+1}\circ(S_{j+1}^{-1}C_{j+1,l}^{-1}TC_{j,l}S_jc_{j,r}+S_{j+1}^{-1}C_{j+1,l}^{-1}TC_{j,l}S_jC_{j,r}\,x$$
$$+S_{j+1}^{-1}(C_{j+1,l}^{-1}TC_{j,l}S_jI_{j,r}+C_{j+1,l}^{-1}I_{j+1,l}))$$

$$=S_{j+1}\circ(c_{j+1,r}+C_{j+1,r}\,x+I_{j+1,r})\quad=S_{j+1}\circ(P_{j+1,r}+I_{j+1,r})$$

If $S_j=I$ for all j, then the symbolic part and the interval part propagate like the interval part in the PE and QR interval methods.

Preconditioning without S_j-matrices

Expression in brackets:

$$\left[C_{j+1,l}^{-1} T C_{j,l}\, x + C_{j+1,l}^{-1} I_{j+1,l}\right] \circ (c_{j,r} + C_{j,r}\, x + I_{j,r})$$

$$= C_{j+1,l}^{-1} T C_{j,l} c_{j,r} + C_{j+1,l}^{-1} T C_{j,l} C_{j,r}\, x + C_{j+1,l}^{-1} T C_{j,l} I_{j,r} + C_{j+1,l}^{-1} I_{j+1,l}$$

$$= c_{j+1,r} + C_{j+1,r}\, x + I_{j+1,r} \quad = P_{j+1,r} + I_{j+1,r}$$

The symbolic part and the interval part propagate like the interval part in the PE and QR interval methods.

Linear Example 1: Pure Contraction

$$B = \begin{pmatrix} -0.6875 & -0.1875 & 0.08838834762 \\ -0.1875 & -0.6875 & 0.08838834762 \\ 0.08838834762 & 0.08838834762 & -0.875 \end{pmatrix} \approx \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

	t	Steps	Accuracy	Time
AWA pe	75.5	463	1.4E+4	M^{-1} failed;
AWA ivpe	74.5	479	6.3E-15	M^{-1} failed;
AWA QR	1000	6611	5.4E-15	10.5
TM na	100	9428	2.5E+31	573
TM SW	60.4	4423	Failed	—
TM PR	100	572	1.6E-12	62

(Orders: AWA 20, TM 18)

Linear Example 2: Pure Rotation

$$B = \begin{pmatrix} 0 & -0.7071067810 & -0.5 \\ 0.7071067810 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

	t	Steps	Accuracy	Time
AWA pe	1000	1925	3.2E-10	2.7
AWA QR	1000	1925	3.2E-10	2.8
TM na	100	9523	6.3E+36	583
TM SW	100	7582	(as PR)	467
TM PR	100	7582	2.2E-8	517

Linear Example 3: Contraction and Rotation

$$B = \begin{pmatrix} -0.125 & -0.8321067810 & -0.3232233048 \\ 0.5821067810 & -0.125 & 0.6767766952 \\ 0.6767766952 & -0.3232233048 & -0.25 \end{pmatrix} \approx \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

	t	Steps	Accuracy	Time
AWA pe	78	218	5.0E+17	Integration failed
AWA ivpe	118	313	1.1E+17	Integration failed
AWA QR	1000	2683	3.9E-10	4.3
TM na	100	9599	9.1E+44	585
TM SW	100	7742	1.2E+107	485
TM PR	100	7741	1.8E-8	530