High accuracy Hermite approximation for space curves in $\mathbb{R}^d$

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1 Introducing the method

In this talk we describe approximation procedures for curves in $\mathbb{R}^d$ which significantly improve the standard approximation order. These methods are based on the observation that the parametrization of a curve is not unique and can be suitably modified to improve the approximation order.

Let

$$C : t \mapsto (f_1(t), \ldots, f_d(t)) \in \mathbb{R}^d, \quad t \in [0, h]$$

be a regular smooth curve in $\mathbb{R}^d$. We want to approximate $C$ using information at the points 0 and $h$ by a polynomial curve

$$\mathcal{P} : t \mapsto (X_1(t), \ldots, X_d(t)) \in \mathbb{R}^d,$$
where $X_i(t), \ i = 1, \ldots, d$ are polynomials of degree $\leq m$. Furthermore, by a change of variables (replacing $t$ by $\frac{t}{h}$) we may assume that $h = 1$. If we choose for $X_i(t), \ i = 1, \ldots, d$ the piecewise Taylor polynomial of degree $\leq m$, then $P$ approximates $C$ with order $m + 1$, i.e.

$$f_i(t) - X_i(t) = O(t^{m+1}), \ i = 1, \ldots, d.$$ 

2 de Boor, Höllig, Sabin


A better approximation order appeared first for planar curves by generalization of cubic Hermite interpolation yielding $6^{th}$ order accuracy. In addition to position and tangent, the curvature is interpolated at each endpoint of the cubic segments.

Let

$$C : s \rightarrow (f_1(s), f_2(s)) \in \mathbb{R}^2$$

be a planar curve. Let $p(t)$ be a cubic polynomial curve that approximates the curve $C$ using
the conditions:

\[
p(i) = f(s_i), \quad \frac{p'(i)}{|p'(i)|} = \frac{f'(s_i)}{|f'(s_i)|},
\]

\[
\frac{|p'(i) \times p''(i)|}{|p'(i)|^3} = \frac{|f'(s_i) \times f''(s_i)|}{|f'(s_i)|^3},
\]

where \( i = 0, 1 \). Note that the curvature of \( p(t) \) and \( f(s) \) will be the same at the end points \( t = 0, t = 1 \). The polynomial \( p(t) \) is presented in the Bézier Form

\[
p(t) = \sum_{i=0}^{3} b_i B_i^3(t) \quad t \in [0, 1],
\]

where \( B_i^3(t) \) are the Bernstein polynomials, and \( b_i, i = 0, 1, 2, 3 \) denote the Bézier control points. Applying these conditions gives

\[
\begin{align*}
p(0) &= f(s_0) \quad \Rightarrow \quad b_0 = f(s_0) \\
p(1) &= f(s_1) \quad \Rightarrow \quad b_3 = f(s_1) \\
\frac{p'(0)}{|p'(0)|} &= \frac{f'(s_0)}{|f'(s_0)|} \quad \Rightarrow \quad b_1 = b_0 + \frac{|p'(0)|}{3} \frac{f'(s_0)}{|f'(s_0)|}, \\
\frac{p'(1)}{|p'(1)|} &= \frac{f'(s_1)}{|f'(s_1)|} \quad \Rightarrow \quad b_2 = b_3 - \frac{|p'(1)|}{3} \frac{f'(s_1)}{|f'(s_1)|}.
\end{align*}
\]

For the sake of simplicity, we define

\[
d_0 = \frac{f'(s_0)}{3|f'(s_0)|}, \quad d_1 = \frac{f'(s_1)}{3|f'(s_1)|},
\]

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\[ f(s_0) = f_0, \quad f(s_1) = f_1, \]
\[ |p'(0)| = \alpha_0, \quad |p'(1)| = \alpha_1. \]

Thus the equations become
\[ b_0 = f_0, \quad b_3 = f_1, \]
\[ b_1 = b_0 + \alpha_0 d_0, \quad b_2 = b_3 - \alpha_1 d_1. \] (2)

The Bézier control points \( b_1, b_2 \) are determined by two unknown parameters \( \alpha_0, \alpha_1 \).

The curvatures at the end points \( t = 0, t = 1 \) are
\[ \kappa_0 = \frac{|p'(0) \times p''(0)|}{|p'(0)|^3}, \]
\[ \kappa_1 = \frac{|p'(1) \times p''(1)|}{|p'(1)|^3}, \]
where
\[ \kappa_i = \frac{|f'(s_i) \times f''(s_i)|}{|f'(s_i)|^3}, \quad i = 0, 1. \]

Since
\[ p'(0) = 3(b_1 - b_0), \quad p''(0) = 6b_1 - 12b_2 + 6b_3, \]
thus we have
\[ \kappa_0 = \frac{|3(b_1 - b_0) \times (6b_0 - 12b_1 + 6b_2)|}{|3(b_1 - b_0)|}. \]

Thus the equations become
\[ \kappa_0 = \frac{2}{3\alpha_0^2 d_0} \times (b_2 - b_1). \] (3)
Observing that
\[ b_2 - b_1 = (f_1 - f_0) - \alpha_1 d_1 - \alpha_0 d_0, \]
and set \( a = f_1 - f_0 \), thus we get
\[ (d_0 \times d_1)\alpha_1 = (d_0 \times a) - \frac{3}{2} \kappa_0 \alpha_0^2. \]  
(4)

Similar simplification at the other end point \( t = 1 \) gives
\[ (d_0 \times d_1)\alpha_0 = (a \times d_1) - \frac{3}{2} \kappa_1 \alpha_1^2. \]  
(5)

To summarize, we get the following nonlinear quadratic system
\[
\begin{align*}
(d_0 \times d_1)\alpha_1 &= (d_0 \times a) - \frac{3}{2} \kappa_0 \alpha_0^2, \\
(d_0 \times d_1)\alpha_0 &= (a \times d_1) - \frac{3}{2} \kappa_1 \alpha_1^2,
\end{align*}
\]  
(6)

with the unknown parameters \( \alpha_0, \alpha_1 \).

**Theorem 1** If \( f \) is a smooth curve with nonvanishing curvature and
\[ h := \sup_i |f_{i+1} - f_i| \]
is sufficiently small, then positive solutions of the nonlinear system exist and the corresponding \( p(t) \) satisfies
\[ \text{dist}(f(s), p(t)) = \mathcal{O}(h^6). \]  

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3 Example

Consider the circle

\[ C : s \rightarrow (\cos(s), \sin(s)) \in \mathbb{R}^2. \]

We want to find the cubic polynomial approximation \( p(t) \) that satisfies the nonlinear system at the points \( s_0 = 0 \) and \( s_1 = \pi/8, \pi/16, \pi/32 \).

We compute \( p(t) \) at the starting point \( (s_0 = 0, s_1 = \pi/8) \), the other cases are similarly.

To solve the quadratic system we have to compute the following quantities:

\[
\begin{align*}
    d_0 &= \frac{f'(0)}{|f'(0)|} = (0, 1). \\
    d_1 &= \frac{f'((\pi/2))}{|f'((\pi/2))|} = (-0.382683432, 0.9238795327). \\
    a &= f_1 - f_0 = (-0.076120467, 0.3826834324). \\
    \kappa_0 &= \kappa_1 = 1.
\end{align*}
\]

Then the quadratic system becomes

\[
\begin{align*}
    0.382683432 \alpha_0 &= 0.0761204678 - \frac{3}{2} \alpha_1^2, \\
    0.382683432 \alpha_1 &= 0.076120467 - \frac{3}{2} \alpha_0^2.
\end{align*}
\]
Solving this system numerically for the unknowns $\alpha_0$ and $\alpha_1$ yields the solution

$$\alpha_1 = 0.1715093022, \quad \alpha_0 = 0.08361299186.$$  

The Bézier control points $b_i$, $i = 0, 1, 2, 3$ associated with this solution are

- $b_0 = (1, 0)$,
- $b_1 = (1, 0.08361299186)$,
- $b_2 = (0.989513301, 0.224229499)$,
- $b_3 = (0.92387953, 0.38268343)$.

### 4 Rababah: Planar Curves


A conjecture is studied, which generalizes Taylor theorem and achieves the accuracy of $2m$ for planar curves (rather than $m + 1$) in special cases.
Let $\mathcal{C} : t \rightarrow (f(t), g(t)) \in \mathbb{R}^2$, be a regular smooth planar curve. We seek a polynomial curve
\[ \mathcal{P} : t \rightarrow (X(t), Y(t)) \in \mathbb{R}^2, \]
where $X(t), Y(t)$ are polynomials of degree $m$, that approximate the planar curve $\mathcal{C}$ with high accuracy.

**Conjecture:** A smooth regular curve in $\mathbb{R}^2$ can be approximated by a polynomial curve of degree $\leq m$ with order $\alpha = 2m$.

To illustrate the conjecture, assume, without loss of generality, that
\[ (f(0), g(0)) = (0, 0), \]
and
\[ (f'(0), g'(0)) = (1, 0). \]
Hence for small $t$, $f^{-1}$ exist. Thus, the parameter $x = f(t)$ can be chosen as a local parameter for $\mathcal{C}$, i.e.
\[ \mathcal{C} : t \rightarrow x = f(t) \rightarrow (x, \phi(x)) \]
where
\[ \phi(x) = (g \circ f^{-1})(x) \]
Again, since \( X(0) = 0 \), and \( X'(0) > 0 \), the parameter \( x = X(t) \) can be chosen as a local parameter for \( \mathcal{P} \), i.e.
\[ \mathcal{P} : t \to x = X(t) \to (x, \psi(x)), \]
where
\[ \psi(x) = (Y \circ X^{-1})(x). \]
Thus, the parametrization for \( \mathcal{C} \) is given by
\[ \mathcal{C} : t \to X(t) \to (X(t), \phi(X(t))). \]
Hence, the polynomial curve \( \mathcal{P} \) approximates the planar curve \( \mathcal{C} \) with order \( \alpha \in \mathbb{N} \) iff
\[ \phi(X(t)) - Y(t) = \mathcal{O}(t^\alpha), \]
i.e., iff
\[ \left( \frac{d^j}{dt^j} \right) \{\phi(X(t)) - Y(t)\}|_{t=0} = 0, \quad j = 1, \ldots, \alpha - 1, \]
and
\[ X(0) = Y(0) = 0. \]
Assume that \( X'(0) = 1 \), then the system is determined by \( 2m - 1 \) free parameters. The conjecture follows by comparing the number of equations with the number of parameters.

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5 Example: Cubic case

To illustrate the conjecture in a special case, a cubic parametrization $\mathcal{P}(t)$ is constructed to achieve the optimal approximation order 6. To this end, the following nonlinear system should be solved:

\[
\begin{align*}
\phi_1 X_1 - Y_1 &= 0, \\
\phi_2 X_1^2 + \phi_1 X_2 - Y_2 &= 0, \\
\phi_3 X_1^3 + 3\phi_2 X_1 X_2 + \phi_1 X_3 &= 0, \\
\phi_4 X_1^4 + 6\phi_3 X_1^2 X_2 + 3\phi_2 X_2^2 + 4\phi_2 X_1 X_3 &= 0, \\
\phi_5 X_1^5 + 10\phi_4 X_1^3 X_2 + 15\phi_3 X_1 X_2^2 + 10\phi_3 X_1^2 X_3 + 10\phi_2 X_2 X_3 &= 0,
\end{align*}
\]

where $\phi_i = \phi_i(X(0))$, $X_i = X_i(0)$, and $Y_i = Y_i(0)$ are the $i^{th}$ derivatives of $\phi$, $X$, and $Y$ respectively. The assumption $X_1 = 1$ reduce the nonlinear system to the form

\[
\begin{align*}
\phi_1 - Y_1 &= 0, \\
\phi_2 + \phi_1 X_2 - Y_2 &= 0, \\
\phi_3 + 3\phi_2 X_2 + \phi_1 X_3 - Y_3 &= 0, \\
\phi_4 + 6\phi_3 X_2 + 3\phi_2 X_2^2 + 4\phi_2 X_3 &= 0, \\
\phi_5 + 10\phi_4 X_2 + 15\phi_3 X_2^2 + 10\phi_3 X_3 + 10\phi_2 X_2 X_3 &= 0,
\end{align*}
\]
This nonlinear system has a solution with some restrictions at the derivatives of $\phi$, the following result shows an improvement of the standard Taylor approximation.

**Theorem 2** For $m > 3$, define

$$n_1 = \begin{cases} 
  n & \text{for } m = 3n \text{ or } 3n + 1, \\
  n + 1 & \text{for } m = 3n + 2.
\end{cases}$$

Then for almost all $(\phi_1, \ldots, \phi_{m+n_1}) \in \mathbb{R}^{m+n_1}$ there is a solution for the first $m + n_1$ equations.

As a second result we show that the conjecture is valid for a set of curves of non-zero measure, for which the optimal approximation order $2m$ is attained. To this end, we view equations $m + 1, m + 2, \ldots, 2m - 1$ as a nonlinear system

$$F(\Phi, V) = \left( \frac{d}{dt} \right)^l \phi(X(t))|_{t=0} = 0, \quad l = m + 1, \ldots, 2m - 1,$$

with $V := (X_2, \ldots, X_m), \quad X_1 := 1, \quad \Phi := (\phi_2, \ldots, \phi_{2m-1})$, and show that this system is solvable in a neighborhood of a particular solution $(\Phi^*, X^*)$. The exact statement is
Theorem 3 Define $X_j^* := 0, \quad j = 2, \ldots, m,$ and 

$$
\phi_j^* := \begin{cases} 
1, & j = m \\
0, & \text{otherwise}
\end{cases}
$$

Then $(\Phi^*, X^*)$ is a solution of $F(\Phi, V) = 0,$ where $X^* := (X_2^*, \ldots, X_m^*)$ and $\Phi^* := (\phi_2^*, \ldots, \phi_{2m-1}^*).$ Moreover, there exists a neighborhood of $\Phi^*$ such that the non-linear system is uniquely solvable.

6 Rababah: Space Curves


In fact, without loss of generality we may assume that $(f_1(0), \ldots, f_d(0)) = (0, \ldots, 0), \quad (f'_1(0), \ldots, f'_d(0)) = (1, 0, \ldots, 0),$ so that for small $t$ we can parameterize $C$ in the form

$$
C : t \mapsto X_1(t) \mapsto (X_1(t), \phi_1(X_1(t)), \phi_2(X_1(t)), \ldots, \phi_{d-1}(X_1(t))) \in \mathbb{R}^d.
$$

Since $f'_1(t) > 0$ on a neighborhood $U$ of 0, and $t \mapsto x = f_1(t)$ defines a diffeomorphism on a neighborhood of the origin of the $x$-axis. Thus,
we can choose $x$ as a local parameter for $C$, and get the equivalent representation

$$C : x \mapsto (x, \phi_1(x), \phi_2(x), \ldots, \phi_{d-1}(x)) \in \mathbb{R}^d,$$

where $\phi_i = f_{i+1} \circ f_1^{-1}$, $i = 1, 2, \ldots, d - 1$. Similarly, since $X_1(0) = 0$ and $X_1'(0) > 0$, thus the analogous is true for $t \mapsto x = X_1(t)$, and there is a second reparametrization $t = X_1^{-1}(x)$ for the parameter $t$ on $\mathcal{P}$, and thus the curve $C$ can be represented in the form

$$C : t \mapsto X_1(t) \mapsto (X_1(t), \phi_1(X_1(t)), \phi_2(X_1(t)), \ldots, \phi_{d-1}(X_1(t))) \in \mathbb{R}^d.$$

Thus, $\mathcal{P}$ approximates $C$ with order $\alpha = \alpha_1 + \alpha_2$; $\alpha_1, \alpha_2 \in \mathbb{N}$, iff the parameterizations $X_i(t)$, $i = 1, \ldots, d$ are chosen such that

$$\phi_i(X_1(t)) - X_{i+1}(t) = \mathcal{O}(t^\alpha), \quad i = 1, \ldots, d - 1$$

i.e. iff for $i = 1, \ldots, d - 1$, we have

$$\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}|_{t=0} = 0; \quad j = 1, \ldots, \alpha_1 - 1,$$

$$\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}|_{t=1} = 0; \quad j = 0, 1, \ldots, \alpha_2 - 1,$$

and

$$X_1(1) = 1, \quad X_1(0) = \cdots = X_d(0) = 0.$$
and derivatives of $X_i$, $i = 1, \ldots, d$ are bounded on $[0,1]$. We choose here $X_i(t) = \sum_{j=0}^{m} a_{i,j} t^j$, $i = 1, \ldots, d$. So, the $j^{th}$ derivative of $X_i(t)$ at $t = 1$ is given by the derivatives of $X_i(t)$ at $t = 0$ as follows

$$X_i^{(j)}(1) = \sum_{k=j}^{m} \frac{X_i^{(k)}(0)}{(k-j)!}, \quad j = 1, 2, \ldots, m, \quad i = 1, \ldots, d,$$

where $X_i^{(j)}(t)$ is the $j^{th}$ derivative of $X_i(t)$. The polynomial approximation $\mathcal{P}$ is determined by $dm - 1$ free parameters $\{a_{1,j}\}_{j=2}^{m}, \{a_{2,j}\}_{j=1}^{m}, \ldots, \{a_{d,j}\}_{j=1}^{m}$ and the number of equations is $(\alpha - 1)(d - 1)$. Comparing the number of parameters with the number of equations leads to the following conjecture for $\alpha$.

**Conjecture:** A smooth regular curve in $\mathbb{R}^d$ can be approximated piecewise at two points by a parameterized polynomial curve of degree $\leq m$ with order $\alpha = (m + 1) + \lfloor (m - 1)/(d - 1) \rfloor$.

The significance of the improvement of the approximation order is relatively low for higher dimen-
sions. Table 2 shows a few values of $d, m$ and the optimal order of approximation $\alpha$ from the conjecture.

<table>
<thead>
<tr>
<th></th>
<th>$m = 3$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 2$</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
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<tr>
<td>3</td>
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<td>4</td>
<td>4</td>
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<td>8</td>
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</table>

Table 2: Order of approximation by polynomial curves of degree $m$ in $\mathbb{R}^d$ based on the conjecture.

7 Main results

In the following Theorem 1, we solve $m + \lfloor (m + 1)/(2d - 1) \rfloor$ equations improving the classical Hermite approximation order by $\lfloor (m + 1)/(2d - 1) \rfloor$. 
Theorem 4 For $i = 1, \ldots, d - 1$, let $\phi_i^{(j)} := \phi_i^{(j)}(0)$, $j = 0, \ldots, m$ and $\phi_i^{(m+j)} := \phi_i^{(j)}(1)$, $j = 1, \ldots, n_1$, $n_1 := [(m+1)/(2d-1)]$. Then under appropriate assumptions on

$$(\phi_1^{(1)}, \ldots, \phi_1^{(m+n_1)}, \phi_2^{(1)}, \ldots, \phi_2^{(m+n_1)}, \ldots, \phi_{d-1}^{(1)}, \ldots, \phi_{d-1}^{(m+n_1)}) \in \mathbb{R}^{(d-1)(m+n_1)}$$

there exist polynomial approximations $t \to (X_1(t), X_2(t), \ldots, X_d(t))$ of degree $\leq m$ approximating the curve $t \to (f_1(t), f_2(t), \ldots, f_d(t)) \in \mathbb{R}^d$ piecewise with order $(m + 1) + n_1$.

As a second result we show that the conjecture is valid for a set of curves of non-zero measure, for which the optimal approximation order $m + 1 + n_2$, $n_2 := [(m - 1)/(d - 1)]$ is attained. To this end, we solve the following system, which is equivalent to (1) for $\alpha = m + 1 + n_2$.

For $i = 1, 3, \ldots, od(d)$,

$$\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{t=0} = 0; \quad j = 1, \ldots, m - 1,$$

$$\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{t=1} = 0; \quad j = 0, 1, \ldots, n_2,$$
and for \( i = 2, 4, \ldots, ev(d) \),

\[
\left( \frac{d}{dt} \right)^j \{ \phi_i(X_1(t)) - X_{i+1}(t) \}_{t=0} = 0; \quad j = 1, \ldots, n_2,
\]

\[
\left( \frac{d}{dt} \right)^j \{ \phi_i(X_1(t)) - X_{i+1}(t) \}_{t=1} = 0; \quad j = 0, 1, \ldots, m - 1,
\]

where \( od(d) := \begin{cases} d, & \text{if } d \text{ is odd} \\ d - 1, & \text{else} \end{cases} \) and \( ev(d) := \begin{cases} d, & \text{if } d \text{ is even} \\ d - 1, & \text{else} \end{cases} \).

We set \( V_1 := (X_1^{(n_2)}(0), \ldots, X_1^{(1)}(0)), \quad V_2 := (X_1^{(n_2)}(1), \ldots, X_1^{(1)}(1)), \) and then combine these systems in one system such that the first \( n_2 \) equations for \( V_1 \) are from the first system (i.e. \( \phi_1(X_1(t)) - X_2(t) = 0 \)) and the second \( n_2 \) equations for \( V_2 \) are from the second system (i.e. \( \phi_2(X_1(t)) - X_3(t) = 0 \)) and so on, into a system of the form \( F(\Phi_1, \Phi_2, \ldots, \Phi_{d-1}, V) \), where \( V \) consists of the elements of \( V_1, V_2 \) i.e.

\[
V := (X_1^{(n_2)}(0), \ldots, X_1^{(1)}(0), X_1^{(n_2)}(1), \ldots, X_1^{(1)}(1)),
\]

and

\[
\Phi_i := \left\{ \begin{array}{l} (\phi_i^{(1)}(0), \ldots, \phi_i^{(m)}(0), \phi_i(1), \phi_i^{(1)}(1), \ldots, \phi_i^{(n_2)}(1)), i=1,3, \ldots, \\
(\phi_i^{(1)}(0), \ldots, \phi_i^{(n_2)}(0), \phi_i(1), \phi_i^{(1)}(1), \ldots, \phi_i^{(m)}(1)), i=2,4, \ldots, \end{array} \right\}
\]

We show that this system is solvable in a neighborhood of a particular solution \((\Phi_1^*, \Phi_2^*, \ldots, \Phi_{d-1}^*, X^*)\).
The exact statement is

**Theorem 5** Define \( X_1^{(j)}(0) = X_1^{(j)}(1) := 0 \), \( j = 1, \ldots, n_2 \), \( X^* = (X_1^{(n_2)}(0), \ldots, X_1^{(1)}(0), X_1^{(n_2)}(1), \ldots, X_1^{(1)}(1)) \), and

\[
\Phi_i^* := \begin{cases} 
\phi_i^{(1)}(1) \neq 0, & \text{other elements}=0, i=1,3,\ldots,od(d) \\
\phi_i^{(1)}(0) \neq 0, & \text{other elements}=0, i=2,4,\ldots,ev(d) 
\end{cases}.
\]

Then \( (\Phi_1^*, \Phi_2^*, \ldots, \Phi_{d-1}, X^*) \) is a solution of \( F(\Phi_1, \Phi_2, \ldots, \Phi_{d-1}, V) = 0 \). Moreover, there exists a neighborhood of \( \Phi_1^*, \Phi_2^*, \ldots, \Phi_{d-1}^* \) such that the non-linear system is uniquely solvable.
References


[13] K. Scherer, Parametric polynomial curves of local approximation order 8, Proc. of conf. in