New approach
to rms emittance optimization in RFQ accelerator∗

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1 Mathematical Control Model

Let the evolution of a particle beam be given by the equation

\[ \frac{dx}{dt} = f(t, x, u), \]  

(1)

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f(t, x, u) + \rho \text{div}_x f(t, x, u) = 0, \]  

(2)

where

\[ f(t, x, u) = f_1(t, x, u) + \int_{M_{t,u}} f_2(t, x, y_t) \rho(t, y_t) \, dy_t; \]  

(3)

with initial conditions

\[ x(0) = x_0 \in \overline{M}_0, \quad \rho(0, x) = \rho_0(x). \]  

(4)

Here \( \overline{M}_0 \) is a compact set in \( \mathbb{R}^n \) of a nonzero measure; \( \rho_0(x) \) is a continuously differentiable function. Further, we assume that \( f_1, f_2, \text{div}_x f_1, \text{div}_x f_2 \) are continuous in all their arguments and have continuous partial derivatives with respect to \( x \) and \( y \).

Let \( u = u(t, \hat{x}), \hat{x} = (x_1, x_2, \ldots, x_k)^* = I_k x, \) \( 1 \leq k \leq n \) there \( I_k \) is the corresponding \( k \times n \) matrix with a unit skew series. Assume that \( K \) is the class of controls composed of the vector functions \( u(t, \hat{x}) \) that are continuous in all their variables, along with partial derivatives with respect to \( x \) up to and including the second order. We also assume that the set \( K \) is convex.

Now suppose that the solutions of Eq.1, Eq.2, that are a collection of the vector functions \( x(t, x_0) \) describing the bundle of trajectories emanating from the set \( M_0, \) and the distribution density of particles \( \rho(t, x(t, x_0, u)) \) along these trajectories are determined and unique on the interval \([0, T]\) (with \( T \) fixed) for any control \( u \) from the class \( K \) with values in the compact set \( U \in \mathbb{R}^r. \)
We now introduce the functional

\[ I(u) = \int_0^T \int_{M_{t,u}} \varphi(t, x_t, \rho(t, x_t)) \, dx_t \, dt + \int_{M_{T,u}} g(x_T, \rho(T, x_T)) \, dx_T, \]

where \( \varphi(t, x, \rho) \in C^1(T_0 \times \Omega \times R^1_t) \), \( g(x, \rho) \in C^1(\Omega \times R^1_t) \) are nonnegative functions, \( T_0 = [0, T] \).

Let us study the functional minimization problem Eq.5 for \( u \in K \).

2. Functional of Variation

Let the admissible controls \( u(t, \hat{x}) \) and \( \bar{u}(t, \hat{x}) \), i.e. controls from the class \( K \), be fixes. Consider control

\[ \tilde{u}(t, \hat{x}) = u(t, \hat{x}) + \sigma (\bar{u}(t, \hat{x}) - u(t, \hat{x})) \]

and

\[ x(t) = x(t, x_0, u) \quad \tilde{x}(t) = x(t, x_0, \tilde{u}) \]

are their associated trajectories satisfying the same initial conditions \( x(0) - \tilde{x}(0) = x_0 \). The distribution density of particles along these trajectories is as follows: \( \rho(t) = \rho(t, x(t)) \), \( \dot{\rho}(t) = \rho(t, \tilde{x}(t)) \).

Consider the mapping

\[ \tilde{x}_t = \tilde{x}(x_t) \]

of the set \( M_{t,u} \) into the set \( M_{t,\bar{u}} \) determined by the trajectories Eq.7. It can be easily shown that the Jacobian matrix \( \partial \tilde{x}_t / \partial x_t \) of transformation Eq.8 is definite and continuous on the cross-sections of \( M_{t,u} \).

We have

\[ \det \left( \frac{\partial \tilde{x}_t}{\partial x_t} \right) = 1 + \text{div} \delta x(x_t) + o(\sigma). \]
In this case
\[
\frac{d\delta x}{dt} = \frac{\partial f}{\partial x} \bigg|_\Pi \delta x + \int_{M_{t,u}} \frac{\partial f_2(t, x, y_t)}{\partial y} \delta x(y_t) \rho(t, y_t) \, dy_t + \sigma \frac{\partial f_1}{\partial u}(\bar{u} - u),
\]
(10)

\[
\frac{d\text{div} \delta x}{dt} = \frac{\partial \text{div} f}{\partial x} \bigg|_\Pi \delta x + \int_{M_{t,u}} \frac{\partial \text{div}_x f_2(t, x(t), y_t)}{\partial y} \rho(t, y_t) \delta x(y_t) \, dy_t +
\]
\[
\sigma \left[ \frac{\partial f_1}{\partial u}(\bar{u} - u) + \sum_{i=1}^n \frac{\partial f_{1i}}{\partial u} \left( \frac{\partial \bar{u}}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \right].
\]
(11)

Here
\[
\frac{\partial f}{\partial x} \bigg|_\Pi = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial x}, \quad \text{div} f = \text{Sp} \left( \frac{\partial f}{\partial x} \bigg|_\Pi \right),
\]
\[
\frac{\partial \text{div} f}{\partial x} \bigg|_\Pi = \frac{\partial \text{div} f}{\partial x} + \frac{\partial \text{div} f}{\partial u} \frac{\partial u}{\partial x} + \sum_{i=1}^n \frac{\partial f_{1i}}{\partial u} \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_i}.
\]

It can be shown that
\[
\Delta I = \delta I + o(\sigma),
\]
(12)

where \(o(\sigma)/\sigma \xrightarrow[\sigma \to 0]{} 0\) and
\[
\delta I = \int_0^T \int_{M_{t,u}} \left[ \frac{\partial \varphi}{\partial x} \delta x + \left( \varphi - \frac{\partial \varphi}{\partial \rho} \rho \right) \text{div} \delta x \right] \, dx_t dt +
\]
\[
+ \int_{M_{t,u}} \left[ \frac{\partial g}{\partial x} \delta x + \left( g - \frac{\partial g}{\partial \rho} \rho \right) \text{div} \delta x \right] \, dx_T.
\]

We introduce on the bundle of trajectories the vector function \(\Psi(t, x)\) and function \(\lambda(t, x)\) satisfying along the trajectories the
equations

\[
\frac{d\Psi}{dt} = - \left( \frac{\partial f}{\partial x} \bigg|_\Pi + E \text{div} \ f \right)^* \Psi - \lambda \left( \frac{\partial \text{div} \ f}{\partial x} \bigg|_\Pi \right)^* + \\
\left( \frac{\partial \varphi}{\partial x} \right)^* - \rho \int_{M_{t,u}} \left[ \frac{\partial f_2(t, x(t), y_t)}{\partial x} \Psi(t, y_t) + \left( \frac{\partial \text{div}_x f_2(t, x(t), y_t)}{\partial x} \right)^* \lambda(t, y_t) \right] \, dy_t,
\]

(13)

\[
\frac{d\lambda}{dt} = -\lambda \text{div} \ f + \varphi - \frac{\partial \varphi}{\partial \rho} \rho
\]

(14)

with terminal conditions

\[
\Psi^*(T, x(T)) = -\frac{\partial g(T, x(T), \rho(T, x(T)))}{\partial x},
\]

(15)

\[
\lambda(T, x(T)) = -g(T, x(T), \rho(T, x(T))) + \frac{\partial g(T, x(T), \rho(T, x(T)))}{\partial \rho} \rho(T, x(T)).
\]

(16)

In terms of these relations for the functional variation we obtain the following representation:

\[
\delta I = -\sigma \int_0^T \int_{M_{t,u}} \left[ \left( \Psi^* \frac{\partial f_1}{\partial u} + \lambda \frac{\partial \text{div} \ f_1}{\partial u} \right)(\bar{u} - u) + \\
\sum_{i=1}^n \frac{\partial f_{1i}}{\partial u} \left( \frac{\partial \bar{u}}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \right] \, dx_t \, dt.
\]

(17)

If

\[
\varphi - \frac{\partial \varphi}{\partial \rho} \rho \equiv 0, \quad g - \frac{\partial g}{\partial \rho} \rho \equiv 0,
\]

(18)
then from equations Eq.14 and Eq.16 follows $\lambda \equiv 0$ and

$$
\frac{d\Psi}{dt} = - \left( \frac{\partial f}{\partial x} \bigg|_{\Pi} + E \text{div } f \right)^* \Psi + \left( \frac{\partial \varphi}{\partial x} \right)^* - \rho \int_{M_{t,u}} \frac{\partial f_2(t, x(t), y_t)}{\partial x} \Psi(t, y_t) \, dy_t,
$$

(19)

In this case, from Eq.17 we have

$$
\delta I = -\sigma \int_0^T \int_{M_{t,u}} \Psi^* \frac{\partial f_1}{\partial u} (\bar{u} - u) \, dx_t dt.
$$

(20)

**Remark 1** If the controls are parameterized, the formulas Eq.17, Eq.20 allow writing out the functional gradient with respect to relevant parameters. By way of example, we consider the case where the control class is made up of functions of the form $u = \sum_{m=1}^{N} \alpha_m \Phi_m(\hat{x})$. The simplicity, the equations Eq.18 are taken to hold. Then it follows from Eq.20 that

$$
\frac{\partial I}{\partial \alpha_m} = - \int_0^T \int_{M_{t,u}} \Psi^* \frac{\partial f_1}{\partial u} \Phi_m(\hat{x}) \, dx_t dt, \quad m = 1, N.
$$

(21)

**Remark 2** Variation Eq.20 and, specifically, Eq.21 allow various directional methods to be constructed for numerical minimization of the functional Eq.5 in the beam control problem having regard to interaction of particles.

In conclusion, we may indicate some further generalization for the functionals given on the sections of the trajectory bundle. Let us consider a functional of the form

$$
I(u) = \Phi(\mu_{k_s}^{(1,1)}, \ldots, \mu_{k_s}^{i,j}, \ldots, \mu_{k_s}^{n,n}),
$$

(22)

where $\Phi$ is a function of the parameters $\mu_{k_s}^{i,j}$ that are the moments
of order $k, s$ for the coordinates $x_i, x_j$ of the vector $x_t$: 

$$
\mu_{ks}^{(i,j)} = \int_{M_{t,u}} (x_i - \bar{x}_i)^k (x_j - \bar{x}_j)^s \rho(t, x_t) dx_t,
$$

(23)

$$
i, j = 1, n, \quad j \geq i.
$$

Here $\bar{x}_i, \ i = 1, n$ are the mean values of the $x_i$ coordinates:

$$
\bar{x}_i = \int_{M_{t,u}} x_i \rho(t, x_t) dx_t.
$$

(24)

Functionals of the form Eq.22 are the functions of the functionals given on the beam sections.