In this paper the method of beam dynamics optimization in alternative phase focusing accelerator is suggested. The problem is considered as non-standard problem of the theory of optimal control in discrete systems. The approach of simultaneous optimization of program. and perturbed motions is developed.

1. Mathematical model of optimization

Let the particle dynamics be given by a difference equation system:

\[
x(k + 1) = f(k) = \begin{cases} 
  f_1(k, x(k), u(k)), & k = 0, 1 \\
  f_2(k, x(k), x(k-1), x(k-2), u(k)), & k = 2, ..., N - 1 
\end{cases}
\]

\[
y(k + 1) = F(k) = \begin{cases} 
  F_1(k, x(k), y(k), u(k)), & k = 0, 1 \\
  F_2(k, x(k), x(k-1), x(k-2), y(k), u(k)), & k = 2, ..., N - 1 
\end{cases}
\]

\[
p(k + 1) = P(k, x(k), x(k-2), y(k+1), y(k), p(k), u(k)), & k = 0, ..., N - 1 
\]

(1)

Here \(x(k)\) is the \(n\)-dimensional phase vector defining longitudinal program motion, \(y(k)\) is the \(m\)-dimensional phase vector defining longitudinal perturbed motion, \(p(k)\) is the \(s\)-dimensional phase vector defining transversal perturbed motions, \(u(k)\) is the \(r\)-dimensional control vector.

Given type of system (1) is determined by kind of discrete equations which can be described as the motion of charged particles in **accelerator with tubes of drift**:

\[
\begin{array}{c}
\begin{array}{c}
 l_1 \quad l_2 \\
 r_1 \quad r_2 \\
 l_3 \quad l_4 \\
 r_{\lambda} \quad r_{\lambda} 
\end{array}
\end{array}
\]

**Longitudinal motion**

Motion in half of the drift–tube

\[
\begin{cases} 
  \beta_\lambda = \beta_\mu \\
  \Phi_\lambda = \Phi_\mu + \frac{2\pi \cdot l_\mu}{\beta_\mu \lambda} 
\end{cases}
\]
Motion in the gap:

\[ \beta_k = \beta_0 \left\{ 1 + \frac{\mu}{\gamma_0^2} \left[ \sin(\varphi_n + \varphi_n) - \sin \varphi_n \right] + \frac{\mu^2}{\gamma_0^4} \left[ \frac{1}{\gamma_0^2} \left[ \cos(\varphi_n + \varphi_n) - \cos \varphi_n + \varphi_n \sin \varphi_n \right] \cos(\varphi_n + \varphi_n) - \frac{3}{2} \beta_k^2 \sin(\varphi_n + \varphi_n) - \sin \varphi_n \right] \right\}. \]

\[ \varphi_n = \varphi_n + \mu \frac{\mu}{\gamma_0^2} \left[ \cos(\varphi_n + \varphi_n) - \cos \varphi_n + \varphi_n \sin \varphi_n \right] - \frac{\mu^2}{\gamma_0^4} \left[ \frac{1}{\gamma_0^2} \left[ \cos(\varphi_n + \varphi_n) - \cos \varphi_n + \varphi_n \sin \varphi_n \right] \sin(\varphi_n + \varphi_n) - \sin \varphi_n \right] - \frac{3}{2} \beta_k^2 \left\{ \frac{1}{2} \varphi_n - \frac{1}{4} \left[ \sin(2(\varphi_n + \varphi_n) - \sin 2\varphi_n) \right] + 2 \sin \varphi_n \left[ \cos(\varphi_n + \varphi_n) - \cos \varphi_n \right] + (\varphi_n + \varphi_n) \sin^2 \varphi_n \right\}. \]

Here \( \mu = \frac{eE}{2\pi mc^2 \beta_k \gamma_0^2}, \) \( l_{k,T} \) – length of the half of the drift–tube, \( \varphi_n = \frac{2\pi l_{k,T}}{\beta_k \lambda}, \) \( l \) – length of the accelerating gap, \( m \) – the mass of a particle (proton), \( \beta, \varphi_n \) – relative velocity of the particle and a phase of the particle correspond to beginning of the element, \( \beta_k, \varphi_k \) – relative velocity of the particle and a phase of the particle at the end of the element, \( \gamma_0 \) – relative energy, \( \lambda \) – wavelength, \( e \) – the charge of the particle, \( E \) – electric intensity in the gap, \( c \) – the velocity of light.

Transversal motion

\[ p_N = \left( \prod_{k=1}^N M_k \right) \cdot p_0 \]

\[ M_{ax} = \begin{pmatrix} 1 & 0 \\ -\frac{eE_0 \cos(\varphi_n)}{2m_0 \gamma_0 \beta_n c} & 1 \end{pmatrix} \]

\[ M_{ax} = Q \begin{pmatrix} \cos(\Omega t) & \frac{1}{\Omega} \sin(\Omega t) \\ -\frac{1}{\Omega} \sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \]

\[ npu(\Omega^2 > 0) \]

\[ M_{dp} = \begin{pmatrix} 1 & l_{nT} \beta_n (\beta c) \\ 0 & 1 \end{pmatrix} \]

\[ M_{dp} = Q \begin{pmatrix} ch(\Omega t) & \frac{1}{\Omega} sh(\Omega t) \\ \frac{1}{\Omega} sh(\Omega t) & ch(\Omega t) \end{pmatrix} \]

\[ npu(\Omega^2 < 0) \]

\[ M_{ax} = \begin{pmatrix} 1 & 0 \\ \frac{eE_0 \cos(\varphi_n)}{2m_0 \gamma_0 \beta_n c} & 1 \end{pmatrix} \]

\[ \Omega^2 = \frac{\pi eE_0 \beta_n c (\cos(\varphi_n) - \cos(\varphi_n))}{m_0 \lambda \gamma_n (\varphi_k - \varphi_n)} \]

\[ \Omega = \sqrt{\left| \Omega^2 \right|} \]

\[ Q = \sqrt{\frac{\gamma_n}{\gamma_k}} t = \frac{\lambda (\varphi_k - \varphi_k)}{2\pi c} \]

We note, that if relative velocity and the phase of a particle are chosen as components of the phase vector \( x(k) \):
\[ x(k) = \begin{pmatrix} \beta \\ \phi \end{pmatrix} \text{ and } p(k) = \begin{pmatrix} r \\ r' \end{pmatrix}, \text{ and } u(k) = \begin{pmatrix} \phi_n \\ \varphi_n \\ E \end{pmatrix}, \]

particle dynamics can be written in the form. On the trajectories of the system the following quality functionals have been chosen:

\[
I_1(u) = \sum_{k=1}^{N} \int_{M_{k,u}} \phi_k(x(k), y_k, u(k))dy_k + \int_{M_{N,u}} g(y_N)dy_N,
\]

\[
I_2 = \sum_{k=1}^{N} \int_{M_{k,u}} dy(k) \int_{\Omega_{k,u}} p_1(k)^2 dp(k),
\]

\[
I = c_1 I_1 + c_2 I_2. \quad (2)
\]

We have considered minimization problem of the functional (2).

**Results**

Fig. 1 Dependence of relative velocity on length of the accelerator with initial vector of control \( u_0 \)

Fig. 2 Dependence of relative velocity on length of the accelerator after optimisation
Fig. 3 Dependence of relative velocity of programmed particle on length of the accelerator with initial vector of control $u_0$.

Fig. 4 Dependence of relative velocity of programmed particle on length of the accelerator after optimisation.

Fig. 5 Radius oscillations along the accelerator with initial vector of control $u_0$.

Fig. 6 Radius oscillations along the accelerator after optimisation.

Fig. 7 Phase of programmed particle in the middle of gap oscillations along the accelerator with initial vector of control $u_0$.

Fig. 8 Phase particle oscillations along the accelerator after optimisation.
The obtained results show the possibility of applying this mathematical technique for dynamics optimization tasks for charged particles in accelerators.

**REFERENCES**