Station Keeping around Halo Orbits and High Order Sensitivity Analysis of DAEs using Differential Algebra

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Motivation

- Space trajectory and space system design is always affected by uncertainties
  - Uncertainties due to navigation systems
  - Uncertainties in modeling the dynamical environment
- Operating conditions will generally differ from the nominal design
- Suitable algorithms must be developed to
  - estimate the effects of the previous uncertainties
  - design control corrections to compensate possible errors
- Differential algebra is applied to:
  - expand the solution of TPBVPs around nominal solutions
  - expand the solution of DAEs w.r.t. uncertain parameters
Outline

- **Notes on Differential Algebra (DA)**
- **High Order expansion of the flow of ODEs**
- **High Order Two-Point Boundary Value Problem (TPBVP) Solver**
  - Lambert problem
  - Station keeping (SK) around Halo orbits
- **High Order Integration of DAEs**
  - Reduction of a DAE to an equivalent implicit ODE
  - High order integration of implicit ODEs based on DA
  - Simple pendulum
- **High Order Sensitivity Analysis of DAEs**
  - Double link manipulator with uncertain viscous friction coefficients
- **Conclusions and Future Work**
Notes on Differential Algebra

- DA is an algebra of Taylor polynomials, which can be readily implemented in a computer environment.

DA enables the automatic computation of the Taylor expansion of any function $f$ of $v$ variables up to the arbitrary order $n$.

- Unlike standard automatic differentiation tool, the analytic operations of differentiation and antiderivation are introduced.

- A DA number can be seen as a Taylor Model without the interval remainder bound:

$$ (P_{\alpha,f}, I_{\alpha,f}) $$

$$ P_{\alpha,f} \in nD_v $$

- $nD_v$ is the DA framework for Taylor polynomials of $v$ variables and order $n$. 
Expansion of ODEs Flows (1/2)

- Consider the ODE initial value problem:
  \[ \dot{x} = f(x), \; x(0) = x_0 \]

- Any integration scheme is based on algebraic operations, involving the evaluation of \( f \) at several integration points.

- Replacing \( x_0 \) with \( [x_0] = (x_0, 1) \) and carrying out all the operations in \( n D_v \) enable the evaluation of the Taylor expansion of the ODE flow.

- Example: explicit Euler’s scheme
  \[ [x]_{k+1} = [x]_k + f([x]_k) \cdot h \]

  At each step, \( [x]_{k+1} \) is the \( n \)-th Taylor expansion of the flow of the ODE.
Expansion of ODEs Flows (2/2)

- Example: 2-Body Problem
  - Eccentricity: 0.5
  - Starting point: pericenter
  - Integration scheme: Störmer/Verlet (order 2 symplectic)
  - Order of the flow expansion: 5

- Sensitivity analysis with respect to the initial conditions
  - An uncertainty box of size 0.01 AU on the initial position is propagated by means of the 5th order expansion of the flow
Expansion of ODEs Flows (2/2)

- Example: 2-Body Problem
  - Eccentricity: 0.5
  - Starting point: pericenter
  - Integration scheme: Störmer/Verlet (order 2 symplectic)
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- Sensitivity analysis with respect to the initial conditions
High Order TPBVP solver (1/3)

- Example: Lambert problem

Given

- initial position: $r_1$
- final position: $r_2$
- time of flight: $tof$

Find the initial velocity, $v_1$, the spacecraft must have to reach $r_2$ in $tof$

- Various algorithms exist to identify a nominal solution of this TPBVP, based on iterative procedures
High Order TPBVP solver (2/3)

- Given a nominal solution, $\bar{v}_i$, to the Lambert problem:
  - Use DA to expand the flow of the ODE w.r.t. $r_i$ and $v_i$:
    $$\begin{pmatrix} \delta r_f \\ \delta r_i \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{r_f} \\ \mathcal{I}_{r_i} \end{pmatrix} \begin{pmatrix} \delta r_i \\ \delta v_i \end{pmatrix}$$
  - Build the following map:
    $$\begin{pmatrix} \delta r_f \\ \delta r_i \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{r_f} \\ \mathcal{I}_{r_i} \end{pmatrix} \begin{pmatrix} \delta r_i \\ \delta v_i \end{pmatrix}$$
  - Invert it:
    $$\begin{pmatrix} \delta r_i \\ \delta v_i \end{pmatrix} = \left( \begin{pmatrix} \mathcal{M}_{r_f} \\ \mathcal{I}_{r_i} \end{pmatrix} \right)^{-1} \begin{pmatrix} \delta r_f \\ \delta r_i \end{pmatrix}$$
  - By imposing $\delta r_f = 0$, the previous map delivers a Taylor series expansion of the solution of the TPBVP in $\delta r_i$. 
High Order TPBVP solver (3/3)

- Given a displacement from the nominal initial position, $\delta r_i$, the evaluation of the previous map delivers the corrections to the nominal initial velocity, $\delta v_i$, to reach the final desired nominal position, $\bar{r}_f$

- Test case: Earth-Mars transfer (Mars Express)
High Order TPBVP solver (3/3)

- Given a displacement from the nominal initial position, $\delta r_i$, the evaluation of the previous map delivers the corrections to the nominal initial velocity, $\delta v_i$, to reach the final desired nominal position, $\tilde{r}_f$.

- Test case: Earth-Mars transfer (Mars Express)

`error box of size 0.1 AU`
High Order TPBVP solver (3/3)

- Given a displacement from the nominal initial position, $\delta r_i$, the evaluation of the previous map delivers the corrections to the nominal initial velocity, $\delta v_i$, to reach the final desired nominal position, $\bar{r}_f$.

- Test case: Earth-Mars transfer (Mars Express)

**no corrections**

![Diagram showing no corrections](image1)

**5-th order corrections**

![Diagram showing 5-th order corrections](image2)
SK around Halo Orbits (1/2)

- Circular restricted three body problem:
  - $m_1, m_2$ move on circular orbits
  - $m_3 << m_1, m_2$

- The Halo orbit is a 3-dimensional periodic solution around $L_1, L_2, L_3$
  - five equilibrium points
Station Keeping on a Halo orbit around L1:
Given a nominal halo orbit, design the correction maneuvers to compensate dynamical perturbations

TPBVP formulation: given a displacement from the current nominal state, cancel the error after a given time

Example:

- Reference halo orbit ($A_z = 8000$ km)
- Uncertainty on spacecraft position at intersection with x-z
- Error cancellation after 0.5 period
SK around Halo Orbits (2/2)

- Station Keeping on a Halo orbit around L1:
  Given a nominal halo orbit, design the correction maneuvers to compensate dynamical perturbations

- TPBVP formulation: given a displacement from the current nominal state, cancel the error after a given time

- Example:
  - Reference halo orbit ($A_z = 8000$ km)
  - Uncertainty on spacecraft position at intersection with x-z
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SK around Halo Orbits (2/2)

- Station Keeping on a Halo orbit around L1:
  Given a nominal halo orbit, design the correction maneuvers to compensate dynamical perturbations

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- Example:
  - Reference halo orbit ($A_z = 8000$ km)
  - Uncertainty on spacecraft position at intersection with x-z
  - Error cancellation after 0.5 period

![3rd order corrections](image)
SK around Halo Orbits (2/2)

- Station Keeping on a Halo orbit around L1:
  Given a nominal halo orbit, design the correction maneuvers to compensate dynamical perturbations

- TPBVP formulation: given a displacement from the current nominal state, cancel the error after a given time

- Example:
  - Reference halo orbit ($A_z = 8000$ km)
  - Uncertainty on spacecraft position at intersection with x-z
  - Error cancellation after 0.5 period

![10th order corrections]
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- Notes on Differential Algebra (DA)
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- High Order Integration of Differential Algebraic Equations (DAEs)
  - Reduction of a DAE to an equivalent implicit ODE
  - High order integration of implicit ODEs based on DA
  - Simple and double pendulums
- High Order Sensitivity Analysis of DAEs
  - Double link manipulator with uncertain viscous friction coefficients
- Conclusions and Future Work
Depth and Contraction

- To any element of $[f] \in nD_v$, define the depth $\lambda([f])$ as:

$$\lambda([f]) = \begin{cases} 
\text{Order of first nonvanishing derivative of } f & \text{if } [f] \neq 0 \\
n + 1 & \text{if } [f] = 0
\end{cases}$$

- Any function $f$ with nonvanishing 0th order has $\lambda([f]) = 0$

- Given an operator $\mathcal{O}$ on the set $M \subset nD_v$, it is said to be **contracting** on $M$ if, for any $a, b \in M$ with $a \neq b$,

$$\lambda(\mathcal{O}(a) - \mathcal{O}(b)) > \lambda(a - b)$$

After the application of $\mathcal{O}$, the derivatives in $a$ and $b$ agree to a higher order than before
Fixed Point Theorem

**Theorem:**
Let \( \mathcal{O} \) be a **contracting operator** on \( M \subset nD_v \) that maps \( M \) into \( M \). Then:

- \( \mathcal{O} \) has a **unique** fixed point \( a \in M \) that satisfies the fixed point problem
  \[
  a = \mathcal{O}(a)
  \]
- The sequence \( a_k = \mathcal{O}(a_{k-1}) \), starting from \( a_0 \in M \) for \( k = 1, 2, \ldots \), converges to the fixed point \( a \) in **finitely many steps**

Suppose a fixed point problem \( a = H(a) \) is to be solved

- Bring the problem into \( nD_v \) \( \Rightarrow a = H(a) \)
- Use the fixed point theorem to converge to the DA solution of it
Reduction of DAEs to Implicit ODEs

- Consider the generalization of the first order ODE problem:
  \[ x' = f(t, x, z) \]
  \[ 0 = g(t, x, z) \]

- An implicit form can be obtained by introducing \( \xi = (x, z)^T \)
  \[ F(t, \xi, \xi') = \begin{pmatrix} x' - f(t, x, z) \\ g(t, x, z) \end{pmatrix} = 0 \]

- However, the resulting Jacobian matrix is not regular:
  \[ \left| \frac{\partial F(t, u, v)}{\partial v} \right| = \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix} = 0 \]

  The regularity assumption of a general implicit ODE problem is not met
Reduction of DAEs to Implicit ODEs

- The most general DAE is readily obtained:

\[
\begin{align*}
  f_1(t, x_1, \ldots, x_1^{(\xi_{11})}, \ldots, x_v, \ldots, x_v^{(\xi_{1v})}) &= 0 \\
  \vdots &= 0 \\
  f_v(t, x_1, \ldots, x_1^{(\xi_{v1})}, \ldots, x_v, \ldots, x_v^{(\xi_{vv})}) &= 0
\end{align*}
\]

- The most common approach to solve a DAE is to differentiate the system until \( v \) equations can be picked up such that the Jacobian of the new system is regular.

The DAE problem is reduced to an implicit ODE problem.
DA Integration of Implicit ODEs

Consider the first order implicit ODE initial value problem:

$$F(t, x, x') = 0, \quad x(t_0) = x_0$$

with regular Jacobian matrix \( \partial F(t, u, v) / \partial v \)

Algorithm for a single \( n \)-th order integration step:

1. Solve \( F(t_0, x_0, x') = 0 \) for a consistent \( x'(t_0) = x'_0 \)

2. Rewrite the original problem in a derivative-free form (\( \xi = x' \)):

   $$\Phi(t, x_0, \xi) = F\left(t, x_0 + \int_{t_0}^{t} \xi(\tau)d\tau, \xi\right) = 0$$

3. Substitute \( \zeta(t) = \xi(t) - x'_0 \) to obtain the origin-preserving form:

   $$\Psi(t, x_0, \zeta) = \Phi(t, x_0, \zeta(t) + x'_0) = 0$$
Consider the Taylor expansion of \( \Psi \) around \((t, \zeta) = (t_0, 0)\). Since \( \Psi(t_0, x_0, 0) = 0 \), we have:

\[
\Psi(t, x_0, \zeta) = L_\zeta(\zeta) + L_R(t) + N(t, \zeta) = 0
\]

Obtain the equivalent fixed point formulation for \( \zeta \):

\[
\zeta(t) = \mathcal{H}(\zeta) = -L_\zeta^{-1}(\Psi(t, x_0, \zeta) - L_\zeta(\zeta))
\]

Using \( \mathcal{H} \), define a sequence \((a_k)\) of DA vectors by \( a_0 = 0 \) and:

\[
a_{k+1} = \mathcal{H}(a_k)
\]

The following statements can be demonstrated:

- \( \mathcal{H} \) is a contracting operator and it has a unique fixed point.
- The fixed point \( a_{n+1} \) is a DA representative for the derivative of the solution:
  \[
a_{n+1} = [x'(t) - x_0']_n
\]
DA Integration of Implicit ODEs

- The algorithm can be generalized to higher order problems
- Example: 2nd order implicit ODE initial value problem

\[ e^{x''} + x'' + x = 0 \]
\[ x(0) = x = 1 \]
\[ x'(0) = x_0' = 0 \]

0.0156 s on a AMD Athlon(tm) 2.01 GHz desktop pc
Simple Pendulum

Constraints:
\[ \Psi_1 \equiv q_1 - L \cos q_3 = 0 \]
\[ \Psi_2 \equiv q_2 - L \sin q_3 = 0 \]

Extended Lagrangian:
\[ L = \frac{1}{2} m (\ddot{q}_1^2 + \ddot{q}_2^2) + \frac{1}{2} I_G \ddot{q}_3^2 - \lambda_1 \Psi_1 - \lambda_2 \Psi_2 \]

Virtual work of external forces:
\[ \delta W = -mg \delta q_2 \]

Resulting system:
\[ m \ddot{q}_1 + \lambda_1 = 0 \]
\[ m \ddot{q}_2 + \lambda_2 + mg = 0 \]
\[ I_G \ddot{q}_3 + \lambda_1 L \sin q_3 - \lambda_2 L \cos q_3 = 0 \]
\[ \ddot{q}_1 + L \cos q_3 \ddot{q}_3^2 + L \sin q_3 \ddot{q}_3 = 0 \]
\[ \ddot{q}_2 + L \sin q_3 \ddot{q}_3^2 - L \cos q_3 \ddot{q}_3 = 0 \]

Parameters value: \[ g = 9.8 \text{ m/s}^2 \quad m = 1 \text{ kg} \quad L = 1 \text{ m} \]
Simple Pendulum

- **Integrator parameters**: step size: 0.1 s; integration order: 16

- **Constraints satisfaction:**

0.2656 s on a AMD Athlon 2.01 GHz desktop pc
Sensitivity Analysis of DAEs using DA

- W.l.g., suppose a sensitivity analysis is of interest on

\[ F(t, x, x', p) = 0, \quad x(t_0) = x_0 \]

with respect to parameter \( p \).

- A single step of the algorithm can be modified to allow the expansion of the solution in time and parameter \( p \):
  - Solve \( F(t_0, x_0, x', p) = 0 \) for a consistent \( x'(t_0, p) = x'_0(p) \)

  \[ \text{Note: suitable DA techniques are available to solve this parametric implicit equation} \]

  - Rewrite the original problem in a derivative-free form

    \[ (\xi(t, p) = x'(t, p) ) : \]

    \[ \Phi(t, x_0, \xi, p) = F \left( t, x_0 + \int_{t_0}^{t} \xi(\tau, p) d\tau, \xi, p \right) = 0 \]

  - ... 

- The fixed point of operator \( \mathcal{H} \), \( a_{n+1} \) is a Taylor expansion of the solution w.r.t. time and \( p \)
Controlled Double-Link Manipulator

- Same physical parameters as the simple pendulum
- Motion on the horizontal plane
- Torques $C_A$ and $C_B$ acting on joints A and B:
  
  $C_A[Nm] = 0.25 \cdot \sin \left( 2\pi \cdot \frac{3}{20} t \right)$
  
  $C_B[Nm] = \frac{0.25}{2} \cdot \left( \cos \left( 2\pi \cdot \frac{1}{10} t \right) - 1 \right)$

- Viscous friction acting at joints A and B with coefficients:
  
  $c_{v,A} = c_{v,B} = 0.03 \text{ Nms}$
Controlled Double-Link Manipulator

- **Initial conditions:**
  \[ q_3(0) = q_6(0) = 0 \quad \text{and} \quad \dot{q}_3(0) = \dot{q}_6(0) = 0 \]

- **Integration interval:** \( 0 < t < 10 \text{ s} \)

- **Step Size:** 0.1 s

- **Integration order:** 10

0.3438 s on a AMD Athlon 2.01 GHz desktop pc
10% uncertainty introduced on $c_{v,A}$ and $c_{v,B}$:

$c_{v,A} \in [c_{v,A}^0 \cdot (1 - 0.1), c_{v,A}^0 \cdot (1 + 0.1)]$

$c_{v,B} \in [c_{v,B}^0 \cdot (1 - 0.1), c_{v,B}^0 \cdot (1 + 0.1)]$

A uniform grid of 121 points has been settled on the previous intervals in the space $c_{v,A} - c_{v,B}$

121, 10-th order, point integrations have been carried out
Uncertainty on Friction Coefficients

- Uncertainty intervals have been represented as additional DA variables
- A 10-th order sensitivity analysis was carried out
Uncertainty on Friction Coefficients

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Uncertainty on Friction Coefficients

- Uncertainty intervals have been represented as additional DA variables
- A 10-th order sensitivity analysis was carried out
- Error on the final position and velocity of point C:

24.125 s on a AMD Athlon 2.01 GHz desktop pc
Conclusions and Future Works

- **Conclusions**
  - High order corrections maneuvers can be designed using the high order TPBVP solver
  - A high order time integration scheme for DAEs has been implemented based on differential algebra
  - The use of DA techniques allows to expand the solution w.r.t. initial conditions and dynamical model parameters
  - The previous algorithms can be effectively used to both analyze and manage uncertainties and errors

- **Future Works**
  - Integration error estimation based on Taylor coefficients analysis
  - Development of suitable laws to vary step size and order
  - Different expansion orders for time and parameters
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The algorithm can be generalized to higher order problems:

- Consider the following 2nd order implicit ODE:
  \[ G(t, x, x', x'') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = x'_0 \]

- Substitute \( \xi = x'' \) to obtain:
  \[
  \Phi(t, \xi) = G \left( t, x_0 + \int_{t_0}^{t} \left( x'_0 + \int_{t_0}^{\tau} \xi(\sigma) d\sigma \right) d\tau, x'_0 + \int_{t_0}^{\tau} \xi(\tau) d\tau, \xi \right)
  \]
  and the previous algorithm works with minor adjustment

- The previous argument can be generalized to higher order implicit ODE problems